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LETTER TO THE EDITOR

Moving mirrors and the black-body spectrum

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Abstract

The Davis–Fulling model (Fulling and Davies 1976 *Proc. R. Soc. Lond.* A **348** 393; Davies and Fulling 1977 *Proc. R. Soc. Lond.* A **356** 237) is studied in the case of a perfect mirror starting from rest, accelerating for a large but finite time T along the trajectory $z(t) = -\ln\cosh(t)$, and after time T moving with constant velocity. In this situation (a mirror with an asymptotically inertial trajectory), the 'in' and 'out' states are well defined and thus the average number of produced particles can be calculated using the Bogolubov coefficients. In this letter, we compute rigorously the Bogolubov coefficient $\beta_{\omega,\omega'}$, and we prove that the black-body spectrum is obtained in the case $1 \sim \omega \ll \omega' \ll T$. The methods used by other authors to obtain the black-body spectrum are also discussed. Finally, we prove that the number of produced particles in the ω mode per unit time is

$$\frac{1}{\pi}\frac{1}{e^{2\pi\omega}-1}.$$

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1. Introduction

The Davies–Fulling model [2, 3] describes the creation of massless particles by a perfect mirror following prescribed trajectories in two dimensions. In this letter, we study the particle production in the case of a perfect mirror starting from the rest, accelerating for a large but finite time T along the trajectory $z(t) = -\ln\cosh(t)$, and after time T moving in an inertial trajectory.

The interest of this two-dimensional problem is due to its simplicity and to the fact that the four-dimensional calculation of the radiation emitted from a collapsing black hole given by Hawking in [1] presents the same features that the model exposed in this letter (see [4]). Then, since the trajectory used in this letter simulates the black-hole collapse better than the trajectory used by Hawking in [1] (see the last paragraph of section 4), our calculation can clarify the mechanism of the Hawking radiation. Moreover, in contrast with [1, 3, 12], we obtain the 'Hawking effect' without an intrinsic loss of information, i.e., preserving quantum pure states (see [8, 10] for details).

To obtain the radiation emitted from the mirror, the Bogolubov coefficient $\beta_{\omega,\omega'}$ must be calculated. This coefficient has been calculated in some works ([3, 12] etc), but in these calculations, there are several serious mathematical mistakes due to wrong approximations. We believe that these mistakes do not help clarify the radiation process, and for this reason they could induce new mistakes in future investigations about this subject.

In this work, we compute rigorously the coefficient $\beta_{\omega,\omega'}$, and we discuss the approximations used in other works to obtain this coefficient.

One of the main results of the letter is the complete justification of the black-body radiation formula

$$|\beta_{\omega,\omega'}|^2 \approx \frac{1}{2\pi\omega'}\frac{1}{\mathrm{e}^{2\pi\omega}-1},$$

but to deduce it we show that one needs the hypothesis $1 \sim \omega \ll \omega' \ll T$ that, to our knowledge, appears here for the first time, and that we believe is quite relevant.

Therefore, when $T \to \infty$, the number of produced particles in the ω mode (namely $N(\omega)$) diverges logarithmically because $N(\omega) = \int_0^\infty \mathrm{d}\omega' |\beta_{\omega,\omega'}|^2$. For this reason, the physically relevant quantity is the number of particles in the ω mode per unit time. This dimensionless quantity is finite and its value is

$$\frac{1}{\pi}\frac{1}{\mathrm{e}^{2\pi\omega}-1}.$$

The calculation of this quantity is very complicated, but due to its physical importance and to the fact that it has never been computed before, we present a detailed calculation in a mathematical appendix.

The letter is organized as follows. In section 2, we review the general quantum theory of moving mirrors following a prescribed trajectory. In section 3, we prove correctly that the black-body spectrum is obtained from the trajectory described in the abstract. In section 4, we discuss the different methods used in others works to obtain the black-body radiation. Finally, in the mathematical appendix we compute exactly the number of produced particles in the ω mode per unit time.

2. Moving mirrors: general theory

Consider a massless scalar field ϕ in the two-dimensional Minkowski spacetime. Assume that the mirror trajectory is $\mathcal{C}^1$ and it has the following form:

$$g(t) = \begin{cases} 0 & \text{if} \quad t \leqslant 0 \\ -\ln\cosh(t) & \text{if} \quad t \in [0, T] \\ A(t-T) - \ln\cosh(T) & \text{if} \quad t \geqslant T, \end{cases} \tag{1}$$

with $A = -\tanh(T) \approx -1 + 2\mathrm{e}^{-2T}$, because we assume that $T \gg 1$. Finally, we impose the following reflection condition $\phi(t, g(t)) = 0\, \forall t \in \mathbb{R}$.

Introducing the lightlike coordinates $u \equiv t - z$ and $v \equiv t + z$, the Klein–Gordon equation becomes

$$\partial^2_{uv}\phi = 0,$$

with boundary condition $\phi(u, p(u)) = 0$, where $v = p(u)$ is the trajectory of the mirror in the (u, v) variables.

The set of ‘in’ and ‘out’ mode functions is

$$\text{in:}\quad \phi_\omega(u,v)=\frac{\mathrm{i}}{2}\sqrt{\frac{1}{\pi\omega}}(\mathrm{e}^{-\mathrm{i}\omega v}-\mathrm{e}^{-\mathrm{i}\omega p(u)});\qquad \omega>0, \tag{2}$$

$$\text{out:}\quad \bar{\phi}_\omega(u,v)=\frac{\mathrm{i}}{2}\sqrt{\frac{1}{\pi\omega}}(\mathrm{e}^{-\mathrm{i}\omega f(v)}-\mathrm{e}^{-\mathrm{i}\omega u});\qquad \omega>0, \tag{3}$$

where we have introduced the function $f(v)=p^{-1}(v)$.

Remark 2.1. In our case the function f has the following form:

$$f(v)=\begin{cases} v & \text{if}\quad v\leqslant 0\\ -\ln(2-\mathrm{e}^{v}) & \text{if}\quad v\in[0,\bar{z}]\\ \frac{1-A}{1+A}v+\frac{2}{1+A}(AT+\ln\cosh(T)) & \text{if}\quad v\geqslant\bar{z},\end{cases}$$

where $\bar{z}=T-\ln\cosh(T)\approx\ln(2)-\mathrm{e}^{-2T}$.

It is a well-known fact that the average number of particles in the ω mode produced from the vacuum, i.e. the density of produced particles per unit of frequency, is (see [6, 11])

$$N(\omega)=\int\mathrm{d}\omega'|\beta_{\omega,\omega'}|^2, \tag{4}$$

where

$$\beta_{\omega,\omega'}\equiv\mathrm{i}\int_{g(t)}^{\infty}\bar{\phi}_\omega(t,z)\overleftrightarrow{\partial_t}\phi_{\omega'}(t,z)\,\mathrm{d}z, \tag{5}$$

is the β-Bogolubov coefficient. Then, in order to obtain the black-body spectrum we must compute $\beta_{\omega,\omega'}$ for the trajectory described by the function $g(t)$.

To compute the Bogolubov coefficient we use formula (5) at time $t=0$. (Note that this coefficient is time independent.) It is easy to check that

$$\beta_{\omega,\omega'}=\frac{\mathrm{i}}{2\pi}\int_0^\infty\left[\sqrt{\frac{\omega'}{\omega}}(\mathrm{e}^{-\mathrm{i}\omega f(z)}-\mathrm{e}^{\mathrm{i}\omega z})\sin(\omega' z)-\sqrt{\frac{\omega}{\omega'}}(f'(z)\,\mathrm{e}^{-\mathrm{i}\omega f(z)}-\mathrm{e}^{\mathrm{i}\omega z})\sin(\omega' z)\right]\mathrm{d}z. \tag{6}$$

Remark 2.2. Note that $\beta_{\omega,\omega'}$ must be understood in the following way $\beta_{\omega,\omega'}\equiv\lim_{\epsilon\to0}\beta_{\omega,\omega'}(\epsilon)$, where

$$\beta_{\omega,\omega'}(\epsilon)=\frac{\mathrm{i}}{2\pi}\int_0^\infty\left[\sqrt{\frac{\omega'}{\omega}}(\mathrm{e}^{-\mathrm{i}\omega f(z)}-\mathrm{e}^{\mathrm{i}\omega z})\sin(\omega' z)\right.$$
$$\left.-\sqrt{\frac{\omega}{\omega'}}(f'(z)\,\mathrm{e}^{-\mathrm{i}\omega f(z)}-\mathrm{e}^{\mathrm{i}\omega z})\sin(\omega' z)\right]\mathrm{e}^{-\epsilon z}\,\mathrm{d}z.$$

Finally, we can make the decomposition $\beta_{\omega,\omega'}=b^1_{\omega,\omega'}+b^2_{\omega,\omega'}+b^3_{\omega,\omega'}$ with

$$b^1_{\omega,\omega'}=-\frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}}\int_0^{\bar{z}}\mathrm{e}^{-\mathrm{i}(\omega f(z)+\omega' z)}\,\mathrm{d}z \tag{7}$$

$$b^2_{\omega,\omega'}=\frac{-1}{4\pi\mathrm{i}}\frac{1}{\sqrt{\omega\omega'}}\,\mathrm{e}^{-\mathrm{i}\omega f(\bar{z})}\,\mathrm{e}^{-\mathrm{i}\omega'\bar{z}}\left\{1+\left(\omega'-\omega\frac{1-A}{1+A}\right)\xi\left(\omega'+\omega\frac{1-A}{1+A}\right)\right\} \tag{8}$$

$$b^3_{\omega,\omega'}=\frac{1}{4\pi\mathrm{i}}\frac{1}{\sqrt{\omega\omega'}}\{1+(\omega'-\omega)\xi(\omega+\omega')\}, \tag{9}$$

where we have introduced the function

$$\xi(x)\equiv-\mathrm{i}\lim_{\epsilon\to0}\int_0^\infty\mathrm{e}^{\mathrm{i}px}\,\mathrm{e}^{-\epsilon p}\,\mathrm{d}p=\frac{1}{x+\mathrm{i}0}=P\frac{1}{x}-\mathrm{i}\pi\delta(x).$$

3. The black-body spectrum

To obtain the black-body spectrum, we must calculate $\beta_{\omega,\omega'}$ in the following case:

$$1 \sim \omega \ll \omega' \ll \omega T \ll T. \tag{10}$$

In this situation we conclude that

$$b^2_{\omega,\omega'} \approx 0; \qquad b^3_{\omega,\omega'} \approx \frac{1}{2\pi \mathrm{i}} \frac{1}{\sqrt{\omega\omega'}} \tag{11}$$

because $\omega\frac{1-A}{1+A} \approx \omega\, \mathrm{e}^{2T} \gg \omega T \gg \omega'$, and $\omega' \gg \omega$. Now we calculate $b^1_{\omega,\omega'}$; we have

$$b^1_{\omega,\omega'} = -\frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}} \int_0^{\bar{z}} \mathrm{e}^{-\mathrm{i}\omega' z}(2-\mathrm{e}^z)^{\mathrm{i}\omega}\,\mathrm{d}z. \tag{12}$$

Making the change $z = \bar{z} - \rho$, we obtain

$$b^1_{\omega,\omega'} = -\frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}}\,\mathrm{e}^{\mathrm{i}\bar{z}(\omega-\omega')} \int_0^{\bar{z}} \mathrm{e}^{\mathrm{i}\omega'\rho}(2\mathrm{e}^{-\bar{z}} - \mathrm{e}^{-\rho})^{\mathrm{i}\omega}\,\mathrm{d}\rho \equiv -\frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}}\,\mathrm{e}^{\mathrm{i}\bar{z}(\omega-\omega')}\mathcal{I}. \tag{13}$$

Note that, $2\mathrm{e}^{-\bar{z}} = 1 + \mathrm{e}^{-2T}$, then we can write

$$\mathcal{I} = \int_0^{\bar{z}} \mathrm{e}^{\mathrm{i}\omega'\rho}(1 - \mathrm{e}^{-\rho} + \mathrm{e}^{-2T})^{\mathrm{i}\omega}\,\mathrm{d}\rho. \tag{14}$$

Consider now the domain

$$D = \{u \in \mathbb{C}/\,\mathrm{Re}\,u \in [0,\bar{z}], \mathrm{Im}\,u \in [0,\bar{\epsilon}]\}, \text{ with } \bar{\epsilon} \ll 1 \text{ and } \bar{\epsilon}\omega' \gg 1.$$

Note that the function $\mathrm{e}^{\mathrm{i}\omega' u}(1 - \mathrm{e}^{-u} + \mathrm{e}^{-2T})^{\mathrm{i}\omega}$ is analytic in D, and thus we can apply the residue theorem in D. We obtain

$$\mathcal{I} = \mathrm{i}\int_0^{\bar{\epsilon}} \mathrm{e}^{-\omega' s}(1 - \mathrm{e}^{-\mathrm{i}s} + \mathrm{e}^{-2T})^{\mathrm{i}\omega}\,\mathrm{d}s - \mathrm{i}\int_0^{\bar{\epsilon}} \mathrm{e}^{\mathrm{i}\omega'\bar{z}}\mathrm{e}^{-\omega' s}(1 - \mathrm{e}^{-\bar{z}}\,\mathrm{e}^{-\mathrm{i}s} + \mathrm{e}^{-2T})^{\mathrm{i}\omega}\,\mathrm{d}s$$
$$+ \int_0^{\bar{z}} \mathrm{e}^{\mathrm{i}\omega' s}\mathrm{e}^{-\omega'\bar{\epsilon}}(1 - \mathrm{e}^{-\mathrm{i}\bar{\epsilon}}\,\mathrm{e}^{-s} + \mathrm{e}^{-2T})^{\mathrm{i}\omega}\,\mathrm{d}s. \tag{15}$$

The last integral is approximately zero because we assume that $\bar{\epsilon}\omega' \gg 1$. In the second integral that appears in (15), we make the approximation

$$1 - \mathrm{e}^{-\bar{z}}\,\mathrm{e}^{-\mathrm{i}s} + \mathrm{e}^{-2T} \approx 1 - \mathrm{e}^{-\bar{z}} + \mathrm{e}^{-2T} = \mathrm{e}^{-\bar{z}},$$

because the variable s satisfies $0 \leqslant s \leqslant \bar{\epsilon} \ll 1$. Then, we obtain

$$-\mathrm{i}\int_0^{\bar{\epsilon}} \mathrm{e}^{\mathrm{i}\omega'\bar{z}}\,\mathrm{e}^{-\omega' s}(1 - \mathrm{e}^{-\bar{z}}\,\mathrm{e}^{-\mathrm{i}s} + \mathrm{e}^{-2T})^{\mathrm{i}\omega}\,\mathrm{d}s \approx -\frac{\mathrm{i}}{\omega'}\,\mathrm{e}^{-\mathrm{i}\bar{z}(\omega-\omega')}. \tag{16}$$

Finally, in the first integral we approximate $1 - \mathrm{e}^{-\mathrm{i}s}$ by $\mathrm{i}s$, and we obtain

$$\mathrm{i}\int_0^{\bar{\epsilon}} \mathrm{e}^{-\omega' s}(1 - \mathrm{e}^{-\mathrm{i}s} + \mathrm{e}^{-2T})^{\mathrm{i}\omega}\,\mathrm{d}s \approx \mathrm{i}\int_0^{\bar{\epsilon}} \mathrm{e}^{-\omega' s}(\mathrm{i}s + \mathrm{e}^{-2T})^{\mathrm{i}\omega}\,\mathrm{d}s. \tag{17}$$

Making the change $\omega' s = y$ we have

$$\mathrm{i}\int_0^{\bar{\epsilon}} \mathrm{e}^{-\omega' s}(1 - \mathrm{e}^{-\mathrm{i}s} + \mathrm{e}^{-2T})^{\mathrm{i}\omega}\,\mathrm{d}s \approx \left(\frac{\mathrm{i}}{\omega'}\right)^{1+\mathrm{i}\omega} \int_0^{\omega'\bar{\epsilon}} \mathrm{e}^{-y} y^{\mathrm{i}\omega}\,\mathrm{d}y$$
$$\approx \left(\frac{\mathrm{i}}{\omega'}\right)^{1+\mathrm{i}\omega} \Gamma(1+\mathrm{i}\omega), \tag{18}$$

because in (10), we have assumed that $\omega' \ll T$. Consequently, when $1 \sim \omega \ll \omega' \ll T$ we have

$$\beta_{\omega,\omega'} \approx -\frac{\mathrm{i}}{2\pi}\frac{1}{\sqrt{\omega\omega'}}\,\mathrm{e}^{-\mathrm{i}\bar{z}(\omega-\omega')}\left(\frac{\mathrm{i}}{\omega'}\right)^{\mathrm{i}\omega}\Gamma(1+\mathrm{i}\omega). \tag{19}$$

Then, since $|\Gamma(1+\mathrm{i}\omega)|^2 = \pi\omega/\sinh(\pi\omega)$, we obtain the black-body spectrum

$$|\beta_{\omega,\omega'}|^2 \approx \frac{1}{2\pi\omega'}\frac{1}{\mathrm{e}^{2\pi\omega}-1}. \tag{20}$$

Remark 3.1. Formula (20) is also valid in the case $1 \sim \omega \ll \omega' \ll \mathrm{e}^{2T}$.

Remark 3.2. The number of particles produced in the ω mode is finite when T is finite but $N(\omega)$ diverges when $T \to \infty$. In this situation the physically relevant quantity is the number of particles per unit time: $\lim_{T\to\infty}\frac{N(\omega)}{T}$. Note that this quantity is dimensionless. Due to the difficulty of this computation we prove, in the appendix, that for $0 \ll \omega \sim 1$ we have

$$\lim_{T\to\infty}\frac{N(\omega)}{T} = \frac{1}{\pi}\frac{1}{\mathrm{e}^{2\pi\omega}-1}. \tag{21}$$

4. Some comments

1. In [11], the author uses the decomposition $\beta_{\omega,\omega'} = \beta^{I}_{\omega,\omega'} + \beta^{II}_{\omega,\omega'} + \beta^{III}_{\omega,\omega'}$, with

$$\beta^{I}_{\omega,\omega'} = b^{1}_{\omega,\omega'} + \frac{1}{2\pi}\frac{1}{\sqrt{\omega\omega'}}\sin(\omega'\bar{z})\,\mathrm{e}^{-\mathrm{i}\omega f(\bar{z})} \tag{22}$$

$$\beta^{II}_{\omega,\omega'} = b^{2}_{\omega,\omega'} - \frac{1}{2\pi}\frac{1}{\sqrt{\omega\omega'}}\sin(\omega'\bar{z})\,\mathrm{e}^{-\mathrm{i}\omega f(\bar{z})} \tag{23}$$

$$\beta^{III}_{\omega,\omega'} = b^{3}_{\omega,\omega'}. \tag{24}$$

Then in [12], in order to compute $\beta_{\omega,\omega'}$, the author assumes that the mirror accelerates forever, and then he uses the approximation

$$\beta_{\omega,\omega'} \approx \beta^{I}_{\omega,\omega'} + \beta^{III}_{\omega,\omega'},$$

that is, the term $\beta^{II}_{\omega,\omega'}$ is not considered. The author claims: 'The term $\frac{1}{2\pi}\frac{1}{\sqrt{\omega\omega'}}\sin(\omega'\bar{z})\,\mathrm{e}^{-\mathrm{i}\omega f(\bar{z})}$ that appears in $\beta^{I}_{\omega,\omega'}$ oscillates rapidly. Thus, it converges distributionally to zero and it may be neglected.'

Clearly this affirmation is wrong, because the modulus of this term does not tend to zero. The true result, in the approximation $1 \sim \omega \ll \omega' \ll T$, is

$$\left|\beta^{I}_{\omega,\omega'} + \beta^{III}_{\omega,\omega'}\right|^2 \approx \frac{1}{2\pi\omega'}\frac{1}{\mathrm{e}^{2\pi\omega}-1} + \frac{\sin^2(\omega'\ln(2))}{4\pi^2\omega\omega'} \approx \frac{1}{2\pi\omega'}\frac{1}{\mathrm{e}^{2\pi\omega}-1} + \frac{1}{8\pi^2\omega\omega'}.$$

From this result we can deduce that, in order to obtain the black-body spectrum, we need to consider the term $\beta^{II}_{\omega,\omega'}$. And therefore the true approximation of $\beta_{\omega,\omega'}$ is $\beta_{\omega,\omega'} \approx b^{1}_{\omega,\omega'} + b^{3}_{\omega,\omega'}$, because $b^{2}_{\omega,\omega'} \approx 0$.

2. In [3, 4], the authors take the approximation $\beta_{\omega,\omega'} \approx b^{1}_{\omega,\omega'}$, and make the approximation $f(v) = -\ln(\ln(2) - v)$, then a wrong evaluation of $\beta_{\omega,\omega'}$ provides the desired result. More precisely, the authors evaluate essentially the integral

$$\beta_{\omega,\omega'} \approx -\frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}}\lim_{\epsilon\to 0}\int_0^{\ln 2-\mathrm{e}^{-2T}} \mathrm{d}z\,\mathrm{e}^{-\mathrm{i}\omega' z}(\ln(2)-z)^{\mathrm{i}\omega}\,\mathrm{e}^{\epsilon z},$$

in the following way. First making the change $\ln(2) - z = \frac{s}{\omega'}$, we obtain

$$b^{1}_{\omega,\omega'} \approx -\frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}}\frac{2^{-\mathrm{i}\omega'}}{(\omega')^{1+\mathrm{i}\omega}}\lim_{\epsilon\to 0}\int_{\omega'\mathrm{e}^{-2T}}^{\omega'\ln 2}\mathrm{d}s\,\mathrm{e}^{\mathrm{i}s}s^{\mathrm{i}\omega}\,\mathrm{e}^{-\frac{\epsilon}{\omega'}s},$$

then applying a Wick rotation through the residue theorem, we deduce

$$b^1_{\omega,\omega'} \approx -\frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}}\frac{2^{-\mathrm{i}\omega'}}{(\omega')^{1+\mathrm{i}\omega}}\left[\mathrm{i}^{1+\mathrm{i}\omega}\lim_{\epsilon\to 0}\int_0^{\omega'\ln 2}\mathrm{d}s\,\mathrm{e}^{-s+\mathrm{i}\omega\mathrm{e}^{-2T}}(s-\mathrm{i}\omega\,\mathrm{e}^{-2T})^{\mathrm{i}\omega}\,\mathrm{e}^{-\frac{\epsilon}{\omega'}(\mathrm{i}s+\omega\,\mathrm{e}^{-2T})}\right.$$
$$\left. -(\omega'\ln 2)^{1+\mathrm{i}\omega}\lim_{\epsilon\to 0}\int_0^{\frac{\pi}{2}}\mathrm{d}\alpha\,\mathrm{e}^{\mathrm{i}\alpha}\,\mathrm{e}^{\mathrm{i}\omega'\ln(2)\mathrm{e}^{\mathrm{i}\alpha}}\,\mathrm{e}^{-\alpha\omega}\,\mathrm{e}^{-\epsilon ln(2)\,\mathrm{e}^{\mathrm{i}\alpha}}\right].$$

The first integral is approximately $-\frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}}\frac{2^{-\mathrm{i}\omega'}}{(\omega')^{1+\mathrm{i}\omega}}\mathrm{i}^{1+\mathrm{i}\omega}\Gamma(1+\mathrm{i}\omega)$, because $\omega' \gg 1$ and $\omega'\,\mathrm{e}^{-2T}\approx 0$, but a careful analysis shows that the second integral is rather different from zero. The mistake made by the authors in [3] is to approximate this second integral by zero.

The real result, deduced from (11) and (19), is

$$b^1_{\omega,\omega'} \approx -\frac{\mathrm{i}}{2\pi}\frac{1}{\sqrt{\omega\omega'}}\,\mathrm{e}^{-\mathrm{i}\ln(2)(\omega-\omega')}\left(\frac{\mathrm{i}}{\omega'}\right)^{\mathrm{i}\omega}\Gamma(1+\mathrm{i}\omega)-\frac{1}{2\pi\mathrm{i}}\frac{1}{\sqrt{\omega\omega'}},$$

and, clearly, it does not provide the black-body radiation.

3. In the case that the velocity of the mirror is discontinuous, we do not obtain the black-body radiation. For example, if we assume that the velocity of the mirror is zero for $t > T$, that is $A = 0$, we have

$$b^2_{\omega,\omega'} \approx \frac{-1}{2\pi\mathrm{i}}\frac{1}{\sqrt{\omega\omega'}}\,\mathrm{e}^{-\mathrm{i}\omega f(\bar{z})}\,\mathrm{e}^{-\mathrm{i}\omega'\bar{z}},$$

and therefore

$$\left|\beta^I_{\omega,\omega'}\right|^2 \approx \frac{1}{2\pi\omega'}\frac{1}{\mathrm{e}^{2\pi\omega}-1}+\frac{1}{4\pi^2\omega\omega'}.$$

4. To obtain the result in the usual system of units, we take the trajectory

$$g(t) = \begin{cases} 0 & \text{if} \quad t \leqslant 0 \\ -\frac{c}{k}\ln\cosh(kt) & \text{if} \quad t\in[0,T] \\ -c\tanh(kT)(t-T)-\frac{c}{k}\ln\cosh(kT) & \text{if} \quad t \geqslant T, \end{cases} \tag{25}$$

where c is the light speed and k is a constant frequency. Then we have

$$|\beta_{\omega,\omega'}|^2 \approx \frac{1}{2\pi\omega' k}\frac{1}{\mathrm{e}^{2\pi\frac{\omega}{k}}-1} \qquad \text{and} \qquad \lim_{T\to\infty}\frac{N(\omega)}{T} = \frac{1}{\pi}\frac{1}{\mathrm{e}^{2\pi\frac{\omega}{k}}-1}.$$

5. In [6,7] the authors consider trajectories of the type $f(v) = -\ln(\ln(2)-v)$, that we understand as a limit case of the trajectory

$$f(v) = \begin{cases} -\ln(\ln(2)-v) & \text{if} \quad v\in(\infty,\ln(2)-\mathrm{e}^{-2T}) \\ \mathrm{e}^{2T}v-\mathrm{e}^{2T}\ln(2)+1+2T & \text{if} \quad v \geqslant \ln(2)-\mathrm{e}^{-2T}. \end{cases} \tag{26}$$

We consider the case $1 \sim \omega \ll \omega' \ll T$; in this situation an easy calculation provides the following approximation

$$\beta_{\omega,\omega'} \approx -\frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}}\lim_{\epsilon\to 0}\int_{-\infty}^{\ln 2-\mathrm{e}^{-2T}}\mathrm{d}z\,\mathrm{e}^{-\mathrm{i}\omega' z}(\ln(2)-z)^{\mathrm{i}\omega}\,\mathrm{e}^{\epsilon z}$$
$$\approx -\frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}}\frac{2^{-\mathrm{i}\omega'}}{(\omega')^{1+\mathrm{i}\omega}}\lim_{\epsilon\to 0}\int_{\omega'\,\mathrm{e}^{-2T}}^{\infty}\mathrm{d}s\,\mathrm{e}^{\mathrm{i}s}s^{\mathrm{i}\omega}\mathrm{e}^{-\epsilon s}.$$

Then, applying correctly a Wick rotation through the residue theorem, and using that $\omega'\,\mathrm{e}^{-2T}\approx 0$, we obtain

$$\beta_{\omega,\omega'} \approx -\frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}}\frac{2^{-\mathrm{i}\omega'}}{(\omega')^{1+\mathrm{i}\omega}}\,\mathrm{i}^{1+\mathrm{i}\omega}\Gamma(1+\mathrm{i}\omega).$$

Then, we conclude that

$$|\beta_{\omega,\omega'}|^2 \approx \frac{1}{2\pi\omega'}\frac{1}{\mathrm{e}^{2\pi\omega}-1}.$$

Note that this deduction of the black-body spectrum is essentially the same one used by Hawking in [1] to calculate the emitted radiation from a collapsing black hole. It is also interesting to remark that the trajectory (1) simulates the black-hole collapse better than the trajectory (26), although as we have already seen the trajectory (26) provides an easier calculation of the β-Bogolubov coefficient.

5. Conclusions

We have shown that to understand particle creation in the presence of a moving mirror it is crucial to assume that this one is asymptotically inertial (the case of a mirror that accelerates forever must be understood as a limit case). From this assumption, the black-body spectrum can be obtained from a prescribed class of trajectories and it is also possible to calculate explicitly the number of produced particles in the ω mode per unit time. We believe that these results could have applications to study the reaction of the radiated field in the trajectory of the mirror (the back reaction). This is an open question (see [9]), whose answer could be useful to elucidate the naked singularity conjecture (see [4]). We also believe that our results can permit the understanding of particle creation in several cosmological models with event horizons, for example, in the deSitter spacetime (see [5, 13]).

Appendix

In this appendix, we compute the quantity

$$\lim_{T\to\infty}\frac{N(\omega)}{T} = \lim_{T\to\infty}\frac{1}{T}\int_0^\infty \mathrm{d}\omega' |\beta_{\omega,\omega'}|^2.$$

We take the following decomposition:

$$[0,\infty) = [0,\mathrm{e}^{-\sqrt{T}}) \cup \left[\mathrm{e}^{-\sqrt{T}}, T^{\frac{1}{8}}\right) \cup \left[T^{\frac{1}{8}}, \mathrm{e}^{2T}T^{-\frac{1}{8}}\right) \cup \left[\mathrm{e}^{2T}T^{-\frac{1}{8}}, \mathrm{e}^{2T}T^{\frac{1}{8}}\right) \cup \left[\mathrm{e}^{2T}T^{\frac{1}{8}}, \infty\right),$$

and we study the different cases:

1. $\omega' \in [0, \mathrm{e}^{-\sqrt{T}})$
   In this situation we have $b^2_{\omega,\omega'} \approx 0$, $b^3_{\omega,\omega'} \approx 0$ and $\left|b^1_{\omega,\omega'}\right| \leqslant \frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}}\ln 2$. Consequently
   $$\lim_{T\to\infty}\frac{1}{T}\int_0^{\mathrm{e}^{-\sqrt{T}}} \mathrm{d}\omega' |\beta_{\omega,\omega'}|^2 \sim \lim_{T\to\infty}\frac{1}{T}\int_0^{\mathrm{e}^{-\sqrt{T}}} \mathrm{d}\omega'\,\omega' = 0.$$
2. $\omega' \in \left[\mathrm{e}^{-\sqrt{T}}, T^{\frac{1}{8}}\right)$
   Now we have $\left|b^2_{\omega,\omega'}\right| \leqslant \frac{1}{2\pi}\frac{1}{\sqrt{\omega'\omega}}$, $\left|b^3_{\omega,\omega'}\right| \leqslant \frac{1}{2\pi}\frac{1}{\sqrt{\omega'\omega}}$ and $\left|b^1_{\omega,\omega'}\right| \leqslant \frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}}\ln 2$. Thus
   $$\lim_{T\to\infty}\frac{1}{T}\int_{\mathrm{e}^{-\sqrt{T}}}^{T^{\frac{1}{8}}} \mathrm{d}\omega' |\beta_{\omega,\omega'}|^2 \sim \lim_{T\to\infty}\frac{1}{T}\int_{\mathrm{e}^{-\sqrt{T}}}^{T^{\frac{1}{8}}} \mathrm{d}\omega'\left(\frac{1}{\omega'} + \omega'\right) = 0.$$
3. $\omega' \in \left[T^{\frac{1}{8}}, \mathrm{e}^{2T}T^{-\frac{1}{8}}\right)$
   From the remark 2.1, we deduce that in this case formula (20) is valid. Then, we have
   $$\lim_{T\to\infty}\frac{1}{T}\int_{T^{\frac{1}{8}}}^{\mathrm{e}^{2T}T^{-\frac{1}{8}}} \mathrm{d}\omega' |\beta_{\omega,\omega'}|^2 = \frac{1}{\pi}\frac{1}{\mathrm{e}^{2\pi\omega}-1}.$$

4. $\omega' \in \left[\mathrm{e}^{2T} T^{-\frac{1}{8}}, \mathrm{e}^{2T} T^{\frac{1}{8}}\right)$
In this case we have $\left|b^2_{\omega,\omega'}\right| \leqslant \frac{1}{2\pi}\frac{1}{\sqrt{\omega'\omega}}$, $\left|b^3_{\omega,\omega'}\right| \leqslant \frac{1}{2\pi}\frac{1}{\sqrt{\omega'\omega}}$. To compute $b^1_{\omega,\omega'}$ we use formulae (13) and (15) with $\bar\epsilon = 2\mathrm{e}^{-2T} T^{\frac{9}{8}}$. The last integral of (15) is bounded by $\ln(2)\,\mathrm{e}^{-2T}$, the second integral is approximately $-\frac{\mathrm{i}}{\omega'}\,\mathrm{e}^{-\mathrm{i}\bar z(\omega-\omega')}$, and the first integral is approximately

$$\mathrm{i}\int_0^{2\mathrm{e}^{-2T}T^{\frac{9}{8}}} \mathrm{e}^{-\omega' s}(\mathrm{i}s + \mathrm{e}^{-2T})^{\mathrm{i}\omega}\,\mathrm{d}s,$$

making the change $\mathrm{e}^{-2T}u = s$ we obtain

$$(\mathrm{i}\mathrm{e}^{-2T})^{1+\mathrm{i}\omega}\int_0^{2T^{\frac{9}{8}}} \mathrm{e}^{-\omega'\mathrm{e}^{-2T}u}(u-\mathrm{i})^{\mathrm{i}\omega}\,\mathrm{d}u.$$

Note that $\mathrm{e}^{-\omega'\mathrm{e}^{-2T}u} \leqslant \mathrm{e}^{-uT^{-\frac{1}{8}}}$, and $|(u-\mathrm{i})^{\mathrm{i}\omega}| \leqslant \mathrm{e}^{\frac{\pi}{2}\omega}$, thus

$$\left|(\mathrm{i}\mathrm{e}^{-2T})^{1+\mathrm{i}\omega}\int_0^{2T^{\frac{9}{8}}} \mathrm{e}^{-\omega'\mathrm{e}^{-2T}u}(u-\mathrm{i})^{\mathrm{i}\omega}\,\mathrm{d}u\right| \leqslant \mathrm{e}^{-2T}\int_0^{2T^{\frac{9}{8}}} \mathrm{e}^{-uT^{-\frac{1}{8}}}\,\mathrm{d}u \sim T^{\frac{1}{8}}\,\mathrm{e}^{-2T}.$$

From this result, we deduce that

$$\left|b^1_{\omega,\omega'}\right|^2 \sim \frac{1}{\omega'} + \omega'\,\mathrm{e}^{-4T}\left(1 + T^{\frac{1}{4}}\right),$$

consequently

$$\lim_{T\to\infty}\frac{1}{T}\int_{\mathrm{e}^{2T}T^{-\frac{1}{8}}}^{\mathrm{e}^{2T}T^{\frac{1}{8}}} \mathrm{d}\omega' |\beta_{\omega,\omega'}|^2 \sim \lim_{T\to\infty}\frac{1}{T}\left(\ln T + T^{\frac{1}{4}} + T^{\frac{1}{2}}\right) = 0.$$

5. $\omega' \in \left[\mathrm{e}^{2T} T^{\frac{1}{8}}, \infty\right)$
In this case we have

$$b^2_{\omega,\omega'} = \frac{-1}{2\pi\mathrm{i}}\frac{1}{\sqrt{\omega\omega'}}\,\mathrm{e}^{-\mathrm{i}\omega f(\bar z)}\,\mathrm{e}^{-\mathrm{i}\omega'\bar z} + \mathcal{O}\left(\omega'^{-\frac{3}{2}}\right)$$

$$b^3_{\omega,\omega'} = \frac{1}{2\pi\mathrm{i}}\frac{1}{\sqrt{\omega\omega'}} + \mathcal{O}\left(\omega'^{-\frac{3}{2}}\right).$$

To compute $b^1_{\omega,\omega'}$ we use formulae (13) and (15) with $\bar\epsilon = \mathrm{e}^{-T}$. The last integral of (15) is bounded by $\ln(2)\,\mathrm{e}^{-\omega'\mathrm{e}^{-T}}$. The second integral is approximately $-\frac{\mathrm{i}}{\omega'}\,\mathrm{e}^{-\mathrm{i}\bar z(\omega-\omega')} + \mathcal{O}(\omega'^{-2})$. Then when we multiply this second integral times $-\frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}}\,\mathrm{e}^{\mathrm{i}\bar z(\omega-\omega')}$, the first term is cancelled with the first term of $b^3_{\omega,\omega'}$.

The first integral of (15) is approximately

$$\int_0^{\mathrm{e}^{-T}} \mathrm{e}^{-\omega' s}(\mathrm{i}s + \mathrm{e}^{-2T})^{\mathrm{i}\omega}\,\mathrm{d}s = \mathrm{i}\int_0^{\mathrm{e}^{-2T}T^{-\frac{1}{16}}} \mathrm{e}^{-\omega' s}(\mathrm{i}s + \mathrm{e}^{-2T})^{\mathrm{i}\omega}\,\mathrm{d}s$$
$$+\,\mathrm{i}\int_{\mathrm{e}^{-2T}T^{-\frac{1}{16}}}^{\mathrm{e}^{-T}} \mathrm{e}^{-\omega' s}(\mathrm{i}s + \mathrm{e}^{-2T})^{\mathrm{i}\omega}\,\mathrm{d}s \equiv (A) + (B).$$

Note that (A) is approximately

$$\mathrm{i}\int_0^{\mathrm{e}^{-2T}T^{-\frac{1}{16}}} \mathrm{e}^{-\omega' s}\,\mathrm{e}^{-2T\mathrm{i}\omega}\,\mathrm{d}s \approx \frac{\mathrm{i}}{\omega'}\,\mathrm{e}^{-2T\mathrm{i}\omega}\left(1 - \mathrm{e}^{-\omega'\mathrm{e}^{-2T}T^{-\frac{1}{16}}}\right).$$

Then when we multiply (A) times $-\frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}}\,\mathrm{e}^{\mathrm{i}\bar{z}(\omega-\omega')}$, the first term is cancelled with the first term of $b^2_{\omega,\omega'}$.

Finally we study (B). Making the change $s = \mathrm{e}^{-2T}u$, we obtain

$$(B) = (\mathrm{i}\mathrm{e}^{-2T})^{1+\mathrm{i}\omega}\int_{T^{-\frac{1}{16}}}^{\mathrm{e}^T} \mathrm{e}^{-\omega'\mathrm{e}^{-2T}u}(u-\mathrm{i})^{\mathrm{i}\omega}\,\mathrm{d}u,$$

thus

$$|(B)| \leqslant \mathrm{e}^{-2T}\int_{T^{-\frac{1}{16}}}^{\mathrm{e}^T} \mathrm{e}^{-\omega'\mathrm{e}^{-2T}u}\,\mathrm{d}u \leqslant \frac{1}{\omega'}\,\mathrm{e}^{-\omega'\mathrm{e}^{-2T}T^{-\frac{1}{16}}}.$$

With these results, we conclude that

$$\lim_{T\to\infty}\frac{1}{T}\int_{\mathrm{e}^{2T}T^{\frac{1}{8}}}^{\infty}\mathrm{d}\omega'|\beta_{\omega,\omega'}|^2 \sim \lim_{T\to\infty}\frac{1}{T}\left(\int_{\mathrm{e}^{2T}T^{\frac{1}{8}}}^{\infty}\mathrm{d}\omega'\,\omega'\mathrm{e}^{-2\omega'\mathrm{e}^{-T}} + \int_{\mathrm{e}^{2T}T^{\frac{1}{8}}}^{\infty}\mathrm{d}\omega'\frac{1}{\omega'}\,\mathrm{e}^{-2\omega'\mathrm{e}^{-2T}T^{-\frac{1}{16}}}\right).$$

Making the change $v = \omega'\,\mathrm{e}^{-T}$ in the first integral, and the change $v = \omega'\,\mathrm{e}^{-2T}T^{-\frac{1}{16}}$ in the second integral, we deduce that

$$\lim_{T\to\infty}\frac{1}{T}\int_{\mathrm{e}^{2T}T^{\frac{1}{8}}}^{\infty}\mathrm{d}\omega'|\beta_{\omega,\omega'}|^2 \sim \lim_{T\to\infty}\frac{1}{T}\left(\mathrm{e}^{2T}\int_{\mathrm{e}^{T}T^{\frac{1}{8}}}^{\infty}\mathrm{d}v\,v\,\mathrm{e}^{-2v} + \int_{T^{\frac{1}{16}}}^{\infty}\mathrm{d}v\frac{1}{v}\,\mathrm{e}^{-2v}\right) = 0.$$

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